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Sensitivity to initial conditions in stochastic systems

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The time evolution of the mean deviation of initially close trajectories in a stochastic dynamical system is investigated. It is shown both for additive and linearly coupled multiplicative noise that the mean deviation loses its dependence on initial conditions for long times. For shorter times a power law is found for certain types of additive noise processes, in sharp contrast to the exponential separation of initially nearby trajectories in deterministic chaotic systems. Exponential time evolution is obtained for linearly coupled multiplicative noise after an initial transient during which more complex regimes, including a superexponential stage, can take place.

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I. INTRODUCTION

One of the typical signatures of instability of motion exhibited by large classes of dynamical systems is *sensitivity to initial conditions,* whereby initially close trajectories diverge subsequently in the course of time. It is well known that in deterministic chaos the divergence is on the average exponential, its rate being an intrinsic property of the dynamics given by the system's largest Lyapunov exponent. Let

$$\mathbf{X}_{n+1}=\mathbf{f}(\mathbf{X}_n) \tag{1.1}$$

be the evolution law. By definition, the largest Lyapunov exponent is given by

$$\sigma_{\max}=\lim_{n\to\infty}\lim_{|\epsilon|\to 0}\frac{1}{n}\ln\frac{|\mathbf{f}^n(\mathbf{X}_0+\boldsymbol{\epsilon})-\mathbf{f}^n(\mathbf{X}_0)|}{|\boldsymbol{\epsilon}|}, \tag{1.2}$$

where $\boldsymbol{\epsilon}$ represents the initial error vector. Notice the presence of a *double limit* in this definition, to be taken in the precise order indicated, entailing that sensitivity to initial conditions is in essence a global property requiring a full scanning of the tangent space of the reference trajectory. Now, in a typical physical application sensitivity to initial conditions is manifested through the *transient* growth of an initial error, and it is precisely this property that is at the origin of the limited predictability of unstable dynamical systems. It is therefore of interest to extend the formulation of sensitivity to initial conditions based on Eq. (1.2) to account for the time development of a small but finite initial error over a small initial time period, prior to its final saturation to a level determined by the structure of the system's attractor.

In a previous publication [1] C. Nicolis and one of the present authors have put forward such an extended formulation for dissipative systems giving rise to deterministic chaos. The system is run for a certain transient period of time until it reaches its attractor. At this moment, which is regarded as the initial time $n=0$, the state is slightly perturbed by an error vector $\boldsymbol{\epsilon}$, and the evolution of the initial ($\mathbf{X}_0$) and perturbed ($\mathbf{Y}_0=\mathbf{X}_0+\boldsymbol{\epsilon}$) states is followed simultaneously, while at the same time an average over the initial positions on the attractor is performed. This leads to the following equivalent definitions of mean error $\langle u_\epsilon(n)\rangle$,

$$\begin{aligned}\langle u_\epsilon(n)\rangle&=\frac{1}{N}\int_\Gamma d\mathbf{X}_0\rho_0(\mathbf{X}_0)\int_\Gamma d\mathbf{Y}_0|\mathbf{Y}_n-\mathbf{X}_n|\delta(|\mathbf{Y}_0-\mathbf{X}_0|-\epsilon)\\&=\int_\Gamma d\mathbf{X}\int_\Gamma d\mathbf{Y}|\mathbf{Y}-\mathbf{X}|\rho_n(\mathbf{X})\rho_n(\mathbf{Y}),\end{aligned} \tag{1.3}$$

where N is a normalization factor, $\rho_n(\mathbf{X})$ is the probability density on the attractor and Γ denotes the phase space.

In many instances a dynamical system is subjected to a variety of complex perturbations impinging from the environment, which are perceived by the underlying dynamics as *stochastic forcings.* Furthermore, naturally occurring dynamical systems generate their own stochasticity through the mechanism of *thermodynamic fluctuations.* It is therefore of interest to extend the concept of sensitivity to initial conditions to this wider class of systems. This is the principal objective of the present work. Considerable effort has been devoted during recent years in the diagnostics of chaos and, in particular, in distinguishing deterministic chaos and random noise. As we shall see in the present work the properties of these two classes of systems are not substantially different as far as sensitivity to initial conditions is concerned, and in some cases it may even turn out that a noise process is, in a certain well-defined sense, less sensitive (and hence more predictable) than deterministic chaos. It is, therefore, our opinion that the currently overemphasized contrast between noise and chaos should be reassessed.

In Sec. II sensitivity to initial conditions is formulated and explored for systems subjected to additive Gaussian noise. Contrary to deterministic chaos, sensitivity to initial conditions is found to obey to a power law. Multiplicative noise is analyzed in Sec. III. It is found that exponential sensitivity to initial conditions is now generic but can be preceded by more complex regimes, including the possibility of superexponential behavior. The implications of the results and comparisons with recent theories of Lyapunov exponents for stochastic systems are carried out in Sec. IV.

II. GENERAL FORMULATION AND ADDITIVE GAUSSIAN NOISE

Consider a stochastic dynamical system

$$\mathbf{X}_n = \mathbf{f}(\mathbf{X}_{n-1}) + \xi_n, \tag{2.1}$$

where $\mathbf{f}$ represents the deterministic path [drift, Eq. (1.1)] and ξ_n is a (multi) Gaussian Markov noise. Let $\mathbf{X}$ and $\mathbf{Y}$ be two realizations of the process separated at the initial time by a distance ϵ within the range of the finite precision on the initial conditions. Following the discussion in the Introduction on error growth in deterministic systems the mean deviation between these two realizations at a certain time n will be defined as [[1], Eq. (3)]

$$\langle u_\epsilon(n) \rangle = \int_0^\infty u P_{\epsilon,n}(u) du, \tag{2.2a}$$

where the error probability distribution $P_{\epsilon,n}(u)$ is given by

$$P_{\epsilon,n}(u) = \frac{1}{N}\int_\Gamma d\mathbf{X}\int_\Gamma d\mathbf{Y}\,\delta(|\mathbf{X}-\mathbf{Y}|-u)\int_\Gamma d\mathbf{X}_0\int_\Gamma d\mathbf{Y}_0\rho_0(\mathbf{X}_0)\langle\delta(\mathbf{X}-\mathbf{F}^n(\mathbf{X}_0,\{\xi\}))\delta(\mathbf{Y}-\mathbf{F}^n(\mathbf{Y}_0,\{\eta\}))\rangle_{\{\xi\}\{\eta\}}\delta(|\mathbf{Y}_0-\mathbf{X}_0|-\epsilon). \tag{2.2b}$$

Here $\mathbf{F}^n(\mathbf{X}_0,\{\xi\})$ and $\mathbf{F}^n(\mathbf{Y}_0,\{\eta\})$ denote the result of the nth iteration of $\mathbf{X}_n=\mathbf{f}(\mathbf{X}_{n-1})+\xi_n$, $\mathbf{Y}_n=\mathbf{f}(\mathbf{Y}_{n-1})+\boldsymbol{\eta}_n$ from the initial values $\mathbf{X}_0$ and $\mathbf{Y}_0$, respectively. Introducing the transition probabilities $P(\mathbf{X},n|\mathbf{X}_0,0)$, $P(\mathbf{Y},n|\mathbf{Y}_0,0)$ we may further write Eq. (2.2b) in the form

$$P_{\epsilon,t}(u) = \frac{1}{N}\int_\Gamma d\mathbf{X}\int_\Gamma d\mathbf{Y}\,\delta(|\mathbf{X}-\mathbf{Y}|-u)\int_\Gamma d\mathbf{X}_0\int_\Gamma d\mathbf{Y}_0\rho_0(\mathbf{X}_0)P(\mathbf{X},t|\mathbf{X}_0,0)P(\mathbf{Y},t|\mathbf{Y}_0,0)\delta(|\mathbf{Y}_0-\mathbf{X}_0|-\epsilon), \tag{2.3}$$

where we extended the discrete index n to continuous time t. It is convenient to extend formally the range of u to the entire real axis and switch to the characteristic function,

$$G(k,t) \equiv \int_{-\infty}^{\infty} du\, e^{iku} P_{\epsilon,t}(u). \tag{2.4}$$

Inserting Eq. (2.3) into Eq. (2.4) one obtains then for a one variable system

$$G(k,t) = \frac{1}{2}\int_{-\infty}^{\infty} dX_0 \rho_0(X_0)\{G^*(k,X_0,t)G(k,X_0+\epsilon,t) + G^*(k,X_0,t)G(k,X_0-\epsilon,t) + G(k,X_0,t)G^*(k,X_0+\epsilon,t) + G(k,X_0,t)G^*(k,X_0-\epsilon,t)\}, \tag{2.5}$$

where we assumed that the range of variation of X is the entire real axis,

$$G(k,X_0,t) \equiv \int_{-\infty}^{\infty} dX\, e^{ikX} P(X,t|X_0,0), \tag{2.6}$$

and $G^*(k,X_0,t)$ is its complex conjugate. Introducing the cumulant expansion,

$$\langle e^{ikX}\rangle = \exp\left[\sum_{n=1}^{\infty}\frac{(ik)^n}{n!}\langle\langle X^n\rangle\rangle_c\right], \tag{2.7}$$

where $\langle\langle X^n\rangle\rangle_c$ is the nth cumulant, we have for a Gaussian process

$$G(k,t) = 2\int_{-\infty}^{+\infty} dX_0 \rho(X_0)\{\exp[-k^2\sigma^2(t)]\}\cos\left[k\epsilon\frac{\partial}{\partial X_0}\langle X(X_0,t)\rangle\right], \tag{2.8}$$

where $\sigma^2(t) = \langle\langle X^2(t)\rangle\rangle_c$ and $\langle X(X_0,t)\rangle$ is the mean. Hereafter, we will treat the case that the initial position X_0 is delta distributed. The corresponding distribution function is then

$$P_{\epsilon,t}(u) = \left[\frac{1}{4\pi\sigma^2(t)}\right]^{1/2}\left[\exp\left[-\frac{\left[u-\epsilon\frac{\partial}{\partial X_0}\langle X(X_0,t)\rangle\right]^2}{4\sigma^2(t)}\right] + \exp\left[-\frac{\left[u+\epsilon\frac{\partial}{\partial X_0}\langle X(X_0,t)\rangle\right]^2}{4\sigma^2(t)}\right]\right]. \tag{2.9}$$

From Eq. (2.2a), we get for the mean distance

$$\langle u_\epsilon(t)\rangle = 2\left[\frac{\sigma^2(t)}{\pi}\right]^{1/2}\exp\left[-\left[\frac{\epsilon\frac{\partial}{\partial X_0}\langle X(X_0,t)\rangle}{2\sigma(t)}\right]^2\right] + \left[\epsilon\frac{\partial}{\partial X_0}\langle X(X_0,t)\rangle\right]\operatorname{erf}\left[\frac{\epsilon\frac{\partial}{\partial X_0}\langle X(X_0,t)\rangle}{2\sigma(t)}\right], \tag{2.10}$$

where

$$\operatorname{erf} z \equiv (2/\sqrt{\pi})\int_0^z \exp(-t^2)dt \; .$$

For $[\epsilon(\partial/\partial X_0)\langle X(X_0,t)\rangle/2\sigma(t)] \gg 1$ (short times), by introducing the asymptotic expansion of the error function [2]

$$\operatorname{erf}(z) \approx 1 - \frac{2}{\sqrt{\pi}}e^{-z^2}\sum_{m=0}^{\infty}\frac{(-1)^m(2m-1)!!}{2z(2z^2)^m}, \tag{2.11}$$

where $(-1)!!=1$, we have

$$\langle u_\epsilon(t)\rangle \sim \epsilon\frac{\partial}{\partial X_0}\langle X(X_0,t)\rangle . \tag{2.12}$$

For $[\epsilon(\partial/\partial X_0)\langle X(X_0,t)\rangle/2\sigma(t)] \ll 1$ (long times), we find

$$\langle u_\epsilon(t)\rangle \sim 2\left[\frac{\sigma^2(t)}{\pi}\right]^{1/2}, \tag{2.13}$$

where the error function is now expanded as [2]

$$\operatorname{erf}(z) = \frac{1}{\sqrt{\pi}}e^{-z^2}\sum_{m=0}^{\infty}\frac{2^m}{1\times 3\times\cdots\times(2m+1)}z^{2m+1}. \tag{2.14}$$

Equation (2.13) does not depend on ϵ, that is, the system has already lost its memory of initial conditions due to mixing and has reached its asymptotic regime.

As an example, let us consider the Ornstein-Uhlenbeck process,

$$\frac{\partial}{\partial t}X(t) = -\gamma X(t) + R(t), \tag{2.15}$$

where $R(t)$ is a Gaussian white noise,

$$\langle R(t)\rangle = 0, \quad \langle R(t_1)R(t_2)\rangle = \sigma^2\delta(t_1-t_2) . \tag{2.16}$$

For this process one finds straightforwardly

$$\langle X(X_0,t)\rangle = X_0 e^{-\gamma t}, \tag{2.17}$$

and

$$\sigma^2(t) = \frac{\sigma^2}{2\gamma}(1-e^{-2\gamma t}). \tag{2.18}$$

A characteristic time for mixing can be defined by

$$\frac{\epsilon\frac{\partial}{\partial X_0}\langle X(X_0,t^*)\rangle}{2\sigma(t^*)} \sim 1, \tag{2.19}$$

and is found to be

$$t^* = \frac{1}{2\gamma}\ln\left[1+\frac{\gamma\epsilon^2}{2\sigma^2}\right]. \tag{2.20}$$

The behavior of $\langle u_\epsilon(t)\rangle$, Eq. (2.10), will depend critically on the relation between t^* and γ^{-1}, the second characteristic time present in this problem. Two limiting regimes can be distinguished.

A. $t^* \ll \gamma^{-1}$

Using Eq. (2.20) one sees that this inequality amounts to $\frac{1}{2}\ln(1+\gamma\epsilon^2/2\sigma^2) \ll 1$, or

$$\gamma \ll \frac{4\sigma^2}{\epsilon^2}. \tag{2.21}$$

Depending now on the value of time t relative to t^* and γ^{-1} one may simplify Eq. (2.10) using the expansion Equation (2.11) or (2.14). We obtain [Eqs. (2.12) and (2.13)]

$$\langle u_\epsilon(t)\rangle \sim \begin{cases} \epsilon e^{-\gamma t} \quad (t < t^*), & (2.22) \\ \sqrt{2/\gamma\pi}\,\sigma(1-e^{-2\gamma t})^{1/2} \quad (t > t^*). & (2.23) \end{cases}$$

The latter expansion holds uniformly for values of t irrespectively for γ^{-1}, provided that they are larger than t^*. In this range one may distinguish two finer subregimes.

1. $t \ll \gamma^{-1}$

Expanding the exponential in Eq. (2.23) one finds

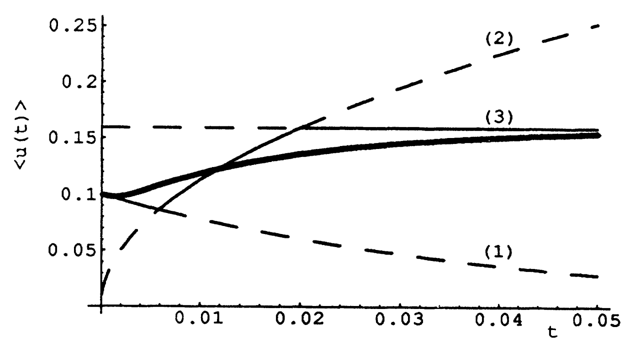

FIG. 1. Short-time evolution of initial distance for additive Gaussian noise. The solid line represents the numerical evaluation of the full equation (2.10) with parameter values $\sigma=1$, $\gamma=25$, and $\epsilon=0.1$. Curve (1) is the analytical curve of Eq. (2.22), (2) is that of Eq. (2.24a), and (3) is the saturation level given by Eq. (2.4b).

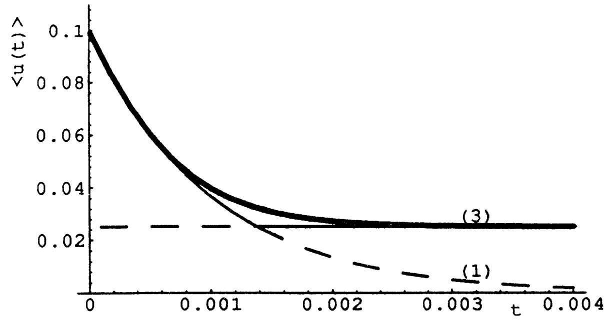

FIG. 2. Same as Fig. 1 but for the larger drift coefficient $\gamma=1000$. Curve (1) is the analytical curve of Eq. (2.22) and (3) is that of Eq. (2.24b).

$$\langle u_\epsilon(t)\rangle \sim \frac{2\sigma}{\sqrt{\pi}} t^{1/2} \quad (t^* < t < \gamma^{-1}), \tag{2.24a}$$

2. $t \gg \gamma^{-1}$

$$\langle u_\epsilon(t)\rangle \sim \sqrt{2/\gamma\pi}\,\sigma \quad (\gamma^{-1} < t) \,. \tag{2.24b}$$

Figure 1 shows the numerical plot of Eq. (2.10) for $\sigma=1$, $\gamma=25$, $\epsilon=0.1$. We also represent in this figure the curves defined by Eqs. (2.22), (2.24a) and (2.24b). We see that there is a short period during which error is damped. Subsequently it starts increasing and eventually it reaches its saturation level. The value of the latter and the time required for reaching it are in good agreement with the theoretical estimates.

B. $t^* \gg \gamma^{-1}$

As γ is increased (or as ϵ is increased at fixed γ and σ) the intermediate regime of power-law behavior [Eq. (2.24a)] shrinks and finally disappears. Initial errors are then damped monotonically until the saturation level is reached, as seen in Fig. 2.

In summary, we have shown that in the presence of stabilizing drift and additive noise errors may at most increase according to a power law rather than exponentially even in the limit of weak damping. This subexponential growth of initial distance agrees with the theory of Lyapunov exponents of stochastic dynamical systems developed by Arnold, where the Lyapunov exponent for linear systems with stable drift and additive noise is zero [3]. In other words, there is no sensitive dependence on initial conditions. This point is in sharp contrast to the exponential instability of trajectories in a large variety of dissipative and conservative dynamical systems giving rise to deterministic chaos, where the sensitivity to initial conditions is regarded as the origin of dynamical randomness [4]. Curiously, as we have also shown in this section, if one takes colored noise as a realization of such randomness, one is led instead to a subexponential separation of nearby trajectories. In this respect, random noise appears to be more predictable than deterministic chaos. We suggest that the connection between deterministic and random dynamical systems ought to be reconsidered in the light of this result.

III. LINEARLY COUPLED MULTIPLICATIVE NOISE

We now consider the case of a multiplicative noise coupled linearly with the system. Let X,Y be two independent realizations starting from different initial conditions. They both obey to the same evolution equation but with different noise realizations,

$$\frac{\partial}{\partial t}X(t)=[\omega_0+\alpha\omega_X(t)]X(t), \tag{3.1}$$

$$\frac{\partial}{\partial t}Y(t)=[\omega_0+\alpha\omega_Y(t)]Y(t), \tag{3.2}$$

$$\langle \omega_i(t)\rangle=0,$$
$$\langle \omega_i(t_1)\omega_j(t_2)\rangle=\langle \omega_1^2\rangle\delta_{ij}e^{-|t_1-t_2|/\tau_c}, \tag{3.3}$$
$$i,j=X \text{ or } Y,$$

where α is a real constant. The explicit solution of Eq. (3.1) is

$$X(t)=X(0)\exp\left[\int_0^t[\omega_0+\alpha\omega_X(t')]dt'\right]. \tag{3.4}$$

Using Stratonovich interpretation for the stochastic differential equations (3.1) and (3.2), we find

$$\langle \ln[|X(t,X_0)|/|X_0|]\rangle=\omega_0 t \tag{3.5}$$

and

$$\langle\langle \{\ln[|X(t,X_0)|/|X_0|]\}^2\rangle\rangle_c=\sigma^2(t), \tag{3.6}$$

where the variance is obtained from Eq. (3.3) in the form [5]

$$\sigma^2(t)=2\alpha^2\langle\omega_1^2\rangle\tau_c^2(t/\tau_c-1+e^{-t/\tau_c}) \tag{3.7}$$

$$\sim\begin{cases}2\alpha^2\langle\omega_1^2\rangle\tau_c t, & \tau_c<t\\ \alpha^2\langle\omega_1^2\rangle t^2, & t<\tau_c.\end{cases} \tag{3.8}$$

Since $\ln[|X(t,X_0)|/|X_0|]$ is a Gaussian process whose average and variance are given by Eqs. (3.5) and (3.6), respectively, the transition probability is [6]

$$P(X,t|X_0,0)=\begin{cases}\dfrac{1}{[2\pi\sigma^2(t)]^{1/2}}\dfrac{1}{X}\exp\left[-\dfrac{(\ln X/X_0-\omega_0 t)^2}{2\sigma^2(t)}\right] & \text{for } X/X_0>0\\ 0, \quad \text{for } X/X_0\le 0.\end{cases} \tag{3.9}$$

Notice that for positive initial values Eq. (3.9) entails that $X(t)$ and $Y(t)$ remain positive and continuous on $[0,\infty)$. From the definition Eqs. (2.2) and (2.3) the deviation between $X(t)$ and $Y(t)$ for this process is now given by

$$\langle u_\epsilon(t)\rangle=\frac{1}{N}\int_0^\infty dX_0\rho_0(X_0)\int_0^\infty dY_0\int_0^\infty dX\,P(X,t|X_0,0)\int_0^\infty dY\,P(Y,t|Y_0,0)|X-Y|\delta(|Y_0-X_0|-\epsilon) \tag{3.10}$$

$$=\frac{1}{N}\int_0^\infty dX\int_0^\infty dY|X-Y|\left\{\int_0^\infty dX_0\rho_0(X_0)P(X,t|X_0,0)P(Y,t|X_0+\epsilon,0)\right.$$
$$\left.+\int_\epsilon^\infty dX_0\rho_0(X_0)P(X,t|X_0,0)P(Y,t|X_0-\varepsilon,0)\right\}. \tag{3.11}$$

This expression can be further written as

$$\langle u_\epsilon(t)\rangle=\frac{1}{N}\int_0^\infty dX\int_0^\infty dY|X-Y|\left\{\int_0^\infty dX_0\rho_0(X_0)P(X,t|X_0+\epsilon,0)P(Y,t|X_0,0)\right.$$
$$\left.+\int_\epsilon^\infty dX_0\rho_0(X_0)P(X,t|X_0-\epsilon,0)P(Y,t|X_0,0)\right\} \tag{3.12}$$

$$=\frac{1}{N}\int_0^\infty dX\left\{\int_0^X dY-\int_X^\infty dY\right\}(X-Y)\left\{\int_0^\infty dX_0\rho_0(X_0)P(X,t|X_0+\epsilon,0)P(Y,t|X_0,0)\right.$$
$$\left.+\int_\epsilon^\infty dX_0\rho_0(X_0)P(X,t|X_0-\epsilon,0)P(Y,t|X_0,0)\right\}. \tag{3.13}$$

Let us introduce the transition probability in terms of transformed variables $x=\ln X$ and $y=\ln Y$,

$$P(X,t|X_0,0)dX=Q(x,t|\ln X_0+\omega_0 t,0)dx, \tag{3.14}$$

where

$$Q(x,t|\ln X_0+\omega_0 t,0)=\frac{1}{[2\pi\sigma^2(t)]^{1/2}}\exp\left[-\frac{(x-\ln X_0-\omega_0 t)^2}{2\sigma^2(t)}\right]. \tag{3.15}$$

Equation (3.13) becomes

$$\langle u_\epsilon(t)\rangle=\frac{1}{2}\int_{-\infty}^\infty dx\left\{\int_{-\infty}^x dy-\int_x^\infty dy\right\}(e^x-e^y)\left\{\int_0^\infty dX_0\rho_0(X_0)Q[x,t|\ln(X_0+\epsilon)+\omega_0 t,0]Q(y,t|\ln X_0+\omega_0 t,0)\right.$$
$$\left.+\int_\epsilon^\infty dX_0\rho_0(X_0)Q[x,t|\ln(X_0-\epsilon)+\omega_0 t,0]Q(y,t|\ln X_0+\omega_0 t,0)\right\}. \tag{3.16}$$

Introducing the identity

$$e^x Q(x,t|\ln X_0+\omega_0 t,0)=X_0\exp[\omega_0 t+\tfrac{1}{2}\sigma^2(t)]Q(x,t|\ln X_0+\omega_0 t+\sigma^2(t),0)$$

and performing the integration over y with the aid of the relation

$$\frac{1}{(2\pi\sigma^2)^{1/2}}\left[\int_{-\infty}^x dy-\int_x^\infty dy\right]\exp\left[-\frac{(y-m)^2}{2\sigma^2}\right]=\operatorname{erf}\left[\frac{x-m}{(2\sigma^2)^{1/2}}\right] \tag{3.17}$$

we arrive at the following expression for $\langle u_\epsilon(t)\rangle$:

$$\langle u_\epsilon(t)\rangle=\frac{1}{N}\exp[\omega_0 t+\tfrac{1}{2}\sigma^2(t)]\int_{-\infty}^\infty dx\left[\int_0^\infty dX_0\rho_0(X_0)(2X_0+\epsilon)Q(x,t|0,0)\operatorname{erf}\left[\frac{x+\ln(1+\epsilon/X_0)+\sigma^2(t)}{[2\sigma^2(t)]^{1/2}}\right]\right.$$
$$\left.+\int_\epsilon^\infty dX_0\rho_0(X_0)(2X_0-\epsilon)Q(x,t|0,0)\operatorname{erf}\left[\frac{x+\ln(1-\epsilon/X_0)+\sigma^2(t)}{[2\sigma^2(t)]^{1/2}}\right]\right]. \tag{3.18}$$

The integration over x can be performed from the identity

$$\int_{-\infty}^\infty dx\,Q(x,t|0,0)\operatorname{erf}\left[\frac{x+z}{[2\sigma^2(t)]^{1/2}}\right]=\operatorname{erf}\left[\frac{z}{2[\sigma^2(t)]^{1/2}}\right], \tag{3.19}$$

which is a direct consequence of [7]

$$\int_0^{\infty} e^{-px-c^2x^2}\operatorname{erf}(cx)dx=\frac{\sqrt{\pi}}{4c}\exp\left[\frac{p^2}{4c^2}\right]\operatorname{erfc}^2\left[\frac{p}{2\sqrt{2}c}\right], \tag{3.20}$$

where $\operatorname{erfc}(y)=1-\operatorname{erf}(y)$. Introducing Eq. (3.19) into Eq. (3.18), we obtain

$$\langle u_\epsilon(t)\rangle=\frac{1}{N}\exp[\omega_0 t+\tfrac{1}{2}\sigma^2(t)]\left[\int_0^{\infty} dX_0\rho_0(X_0)(2X_0+\epsilon)\operatorname{erf}\left[\frac{\ln(1+\epsilon/X_0)+\sigma^2(t)}{2[\sigma^2(t)]^{1/2}}\right]+\int_\epsilon^{\infty} dX_0\rho_0(X_0)(2X_0-\epsilon)\operatorname{erf}\left[\frac{\ln(1-\epsilon/X_0)+\sigma^2(t)}{2[\sigma^2(t)]^{1/2}}\right]\right]. \tag{3.21}$$

We consider the case of deterministic initial conditions,

$$\rho_0(X_0)=\delta(X_0-m),\quad \epsilon<m\ . \tag{3.22}$$

Equation (3.21) reduces to

$$\langle u_\epsilon(t)\rangle=\tfrac{1}{2}\exp[\omega_0 t+\tfrac{1}{2}\sigma^2(t)]\left[(2m+\epsilon)\operatorname{erf}\left[\frac{\ln(1+\epsilon/m)+\sigma^2(t)}{2[\sigma^2(t)]^{1/2}}\right]+(2m-\epsilon)\operatorname{erf}\left[\frac{\ln(1-\epsilon/m)+\sigma^2(t)}{2[\sigma^2(t)]^{1/2}}\right]\right]. \tag{3.23}$$

Furthermore, we assume that the initial state is away from the origin (which is an absorbing state), in the sense, $\epsilon \ll m$. The logarithm in Eq. (3.23) can then be expanded for small ϵ/m, yielding,

$$\langle u_\epsilon(t)\rangle\sim\tfrac{1}{2}\exp[\omega_0 t+\tfrac{1}{2}\sigma^2(t)]\left[(2m+\epsilon)\operatorname{erf}\left[\frac{\epsilon/m+\sigma^2(t)}{2[\sigma^2(t)]^{1/2}}\right]+(2m-\epsilon)\operatorname{erf}\left[\frac{-\epsilon/m+\sigma^2(t)}{2[\sigma^2(t)]^{1/2}}\right]\right]. \tag{3.24}$$

Equation (3.24) exhibits two different behaviors, depending on the value of σ^2 comparatively to ϵ/m:

$$\langle u_\epsilon(t)\rangle\sim\begin{cases}\epsilon\operatorname{erf}\left[\dfrac{\epsilon/m}{2[\sigma^2(t)]^{1/2}}\right]\exp[\omega_0 t+\tfrac{1}{2}\sigma^2(t)], & \sigma^2(t)\ll\dfrac{\epsilon}{m}, \quad (3.25)\\[2ex] 2m\operatorname{erf}\left[\dfrac{[\sigma^2(t)]^{1/2}}{2}\right]^{1/2}\exp[\omega_0 t+\tfrac{1}{2}\sigma^2(t)], & \dfrac{\epsilon}{m}\ll\sigma^2(t). \quad (3.26)\end{cases}$$

We see that the ϵ dependence is dropped out in Eq. (3.26). It remains now to see how $\sigma(t)$ itself scales with ϵ/m and the other characteristic parameters. Upon evaluating the error function in Eqs. (3.25) and (3.26) by asymptotic expansion [Eq. (2.11)] or by series expansion [Eq. (2.14)], one may distinguish between four different characteristic regimes:

$$\langle u_\epsilon(t)\rangle\sim\begin{cases}\text{(a)}\quad \epsilon\exp[\omega_0 t+\tfrac{1}{2}\sigma^2(t)]\quad \left[\sigma^2(t)<\dfrac{\epsilon^2}{4m^2}\right], & (3.27)\\[2ex] \text{(b)}\quad \epsilon\dfrac{\epsilon}{m}\dfrac{1}{[\pi\sigma^2(t)]^{1/2}}\exp[\omega_0 t+\tfrac{1}{2}\sigma^2(t)]\quad \left[\dfrac{\epsilon^2}{4m^2}<\sigma^2(t)<\dfrac{\epsilon}{m}\right], & (3.28)\\[2ex] \text{(c)}\quad 2m\left[\dfrac{\sigma^2(t)}{\pi}\right]^{1/2}\exp[\omega_0 t+\tfrac{1}{2}\sigma^2(t)]\quad \left[\dfrac{\epsilon}{m}<\sigma^2(t)<4\right], & (3.29)\\[2ex] \text{(d)}\quad 2m\exp[\omega_0 t+\tfrac{1}{2}\sigma^2(t)]\quad (4<\sigma^2(t))\ . & (3.30)\end{cases}$$

In the first regime (a), Eq. (3.27) shows exponential sensitivity to initial conditions as long as $\omega_0\geq 0$. If it happens that the drift coefficient is zero, $\omega_0=0$, we obtain from Eq. (3.8)

$$\langle u_\epsilon(t)\rangle\sim\begin{cases}\epsilon\exp[\alpha^2\langle\omega_1^2\rangle\tau_c t]\quad \text{for } t\gg\tau_c, & (3.31)\\ \epsilon\exp[\tfrac{1}{2}\alpha^2\langle\omega_1^2\rangle t^2]\quad \text{for } t\ll\tau_c\ . & (3.32)\end{cases}$$

When the correlation time of noise τ_c is small enough, $\sigma^2(\tau_c) \ll \epsilon^2/4m^2$, we have a purely exponential divergence of nearby initial conditions measured by the quantity $\alpha^2\langle\omega_1^2\rangle\tau_c$, which plays the role of the Lyapunov exponent. On the other hand, for large correlation time such that $\sigma^2(\tau_c) \gg \epsilon^2/4m^2$, Eq. (3.32) shows superexponential sensitivity to initial conditions for values of t limited to the regime (a). Recalling that the dependence on initial distance only appears for the short-time regime, the superexponential instability might become increasingly relevant as the correlation time gets large. Clearly, in this time range the Lyapunov exponent loses its significance. In regime (b), substituting the explicit expressions for the variance Eq. (3.8) into Eq. (3.28), we see that

$$\langle u_\epsilon(t)\rangle \sim \begin{cases} \epsilon\dfrac{\epsilon}{m\alpha}\dfrac{1}{(2\pi\langle\omega_1^2\rangle\tau_c)^{1/2}}\dfrac{1}{\sqrt{t}}\exp[(\omega_0+\alpha^2\langle\omega_1^2\rangle\tau_c)t] \quad \text{for } \dfrac{\epsilon^2}{4m^2} \gg \sigma^2(\tau_c), & (3.33)\\[2ex] \epsilon\dfrac{\epsilon}{m\alpha}\dfrac{1}{(\pi\langle\omega_1^2\rangle)^{1/2}}\dfrac{1}{t}\exp[\omega_0 t+\tfrac{1}{2}\alpha^2\langle\omega_1^2\rangle t^2] \quad \text{for } \dfrac{\epsilon}{m} \ll \sigma^2(\tau_c). & (3.34)\end{cases}$$

The power-law decay correction which now appears reduces the exponential sensitivity to the initial distance. Moreover, Eq. (3.28) depends on the distance from the origin m as well as the initial distance. As we can see from Eq. (3.9) no transitions occur from the origin. Thus, the growth of mean distance would tend to be suppressed as a particle reaches the origin. When the initial condition is chosen far away from the origin this anomalous time regime is reduced, as seen directly from the fact that the interval $\epsilon^2/4m^2 < \sigma^2(t) < \epsilon/m$ decreases monotonically with m/ϵ for $m/\epsilon > 1$.

In the next intermediate regime (c), by substituting Eq. (3.8) into (3.29) it follows that

$$\langle u_\epsilon(t)\rangle \sim \begin{cases} 2m\alpha\left[\dfrac{2\langle\omega_1^2\rangle\tau_c}{\pi}\right]^{1/2}\sqrt{t}\,\exp[(\omega_0+\alpha^2\langle\omega_1^2\rangle\tau_c)t] \quad \text{for } \dfrac{\epsilon}{m} \gg \sigma^2(\tau_c), & (3.35)\\[2ex] 2m\alpha\left[\dfrac{\langle\omega_1^2\rangle}{\pi}\right]^{1/2} t\,\exp[\omega_0 t+\tfrac{1}{2}\alpha^2\langle\omega_1^2\rangle t^2] \quad \text{for } 4 \ll \sigma^2(\tau_c). & (3.36)\end{cases}$$

The exponential growth is now accelerated by contributions in the form of positive powers of t, these powers being 1 for $t<\tau_c$ and $\frac{1}{2}$ for $t>\tau_c$.

Finally in the asymptotic regime (d), Eq. (3.30) shows the same time behavior as in the short-time regime but with a different coefficient. Since the coefficient shows no ϵ dependence, this exponential blow-up is no longer related to the sensitivity to initial conditions but, rather, to the absence of nonlinear saturation terms in the equations of evolution Eqs. (3.1) and (3.2).

Equation (3.23) has been evaluated numerically. Figures 3 and 4 represent $\langle u_\epsilon(t)\rangle$ vs t for $\alpha^2\langle\omega_1^2\rangle=1$, $\tau_c=1$, $\omega_0=0$, $m=1$, and $\epsilon=0.1$. In Fig. 3, analytical curves are deduced from Eqs. (3.27)–(3.30). The numerical plot of the full equation (3.23) starts along an exponential curve but with the superexponent $\frac{1}{2}\alpha^2\langle\omega_1^2\rangle t^2$ as the time range is far below the correlation time of noise $\tau_c=1$ [curve (1)]. The exponential growth is suppressed and $\langle u_\epsilon(t)\rangle$ even starts to decrease in accordance with the theoretical curve (2) of regime (b) at $t\sim 0.05$ which is specified by $\sigma^2(t)\sim\epsilon^2/4m^2$. The plot transits gradually to the intermediate regime where an explosion of the mean distance is observed, in accordance with the theoretical curve (3). In Fig. 4, the explosion continues until $t\sim 3.0$, which is approximately given by the time $\sigma^2(t)\sim 4$ and ends asymptotically with purely exponential growth whose exponent is now given by $\alpha^2\langle\omega_1^2\rangle\tau_c$ again in accordance with theory. We seen that in all time regimes the analytical expressions reproduce the general trends of the numerical plots. It is interesting to note that the initial regime of superexponential sensitivity to initial conditions is well distinguished from the subsequent purely exponential time domain. Such time behavior can be observed as long as the correlation time of noise satisfies the condition $\epsilon^2/4m^2 < \sigma^2(\tau_c)$, which can also be readily achieved by choosing smaller initial error while keeping the correlation time fixed.

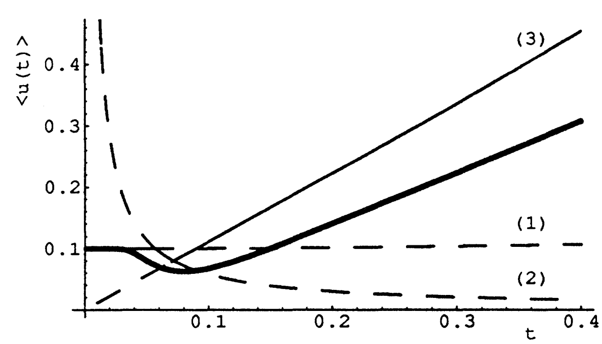

FIG. 3. Short-time evolution of initial distance for linearly coupled multiplicative noise. The solid line represents the numerical evaluation of the full equation (3.23) with parameter value $\alpha^2\langle\omega_1^2\rangle=1$, $\tau_c=1$, $\omega_0=0$, $m=1$, and $\epsilon=0.1$. Curve (1) is the analytical curve of Eq. (3.27), (2) is that of Eq. (3.28), and (3) is that of Eq. (3.29).

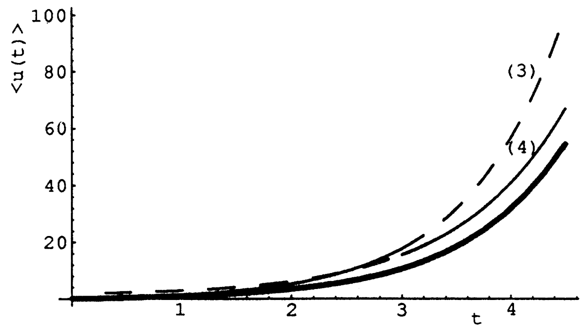

FIG. 4. Same as Fig. 3 but for the longer times. Curve (3) is the analytical curve of Eq. (3.29) and (4) is that of Eq. (3.30).

IV. DISCUSSION

We have outlined a formulation of sensitivity to initial conditions in stochastic systems along very similar lines as in deterministic systems. For additive stochastic perturbations and a linear stable drift part it turned out that error growth follows a power law, entailing that the Lyapunov exponent of this class of systems is zero. For multiplicative stochastic perturbations a variety of regimes are possible for short times, including a superexponential transient stage during which the Lyapunov exponent cannot be defined. All these regimes eventually merge to a regime of exponential growth, but the time beyond which this regime takes over can be arbitrarily delayed depending on the value of the correlation time of the stochastic forcing. If the latter is sufficiently long the exponential regime may be completely masked by the long-time regime where initial errors stabilize (in the mean) to a final saturation value.

Recently Arnold and Kliemann [8] developed an elegant approach to Lyapunov exponents of stochastic systems based on the "thermodynamic" formalism and large deviation theory. They consider the stability properties of average of the pth moment of Eq. (3.4),

$$g(p)\equiv \lim_{t\to\infty}\frac{1}{t}\ln\langle |X(t,X_0)|^p\rangle,\quad g(0)=0,\quad p\in R. \tag{4.1}$$

They show it to be analytic and convex and to control the asymptotic behavior of the probability distribution of $\ln|X(t,X_0)/X_0|$, in the sense that

$$P\left[\frac{1}{t}\ln\frac{|X(t,X_0)|}{|X_0|}\in F\right]\sim\exp[-t\,\inf_{r\in F}I(r)],\quad t\to\infty, \tag{4.2}$$

where $I(r)$ is a Legendre transform of $g(p)$,

$$I(r)=\sup_{p\in R}[rp-g(p)],\quad r\in R. \tag{4.3}$$

Furthermore, for the first and second moments of $\ln[|X(t,X_0)|/|X_0|]$, one obtains

$$\lim_{t\to\infty}\frac{1}{t}\langle\ln[|X(t,X_0)|/|X_0|]\rangle=\frac{\partial}{\partial p}g(p)\bigg|_{p=0} \tag{4.4}$$

(law of large numbers),

$$\lim_{t\to\infty}\frac{1}{t}\langle\{\ln[|X(t,X_0)|/|X_0|]-g'(0)t\}^2\rangle=\frac{\partial^2}{\partial p^2}g(p)|_{p=0} \tag{4.5}$$

(central limit theorem). The first of these relations provides one with the stochastic analog of the Lyapunov exponent.

Let us briefly compare these results with the approach outlined in the present paper. Using the same procedure as in Eqs. (3.5) and (3.6), one can evaluate $g(p)$ explicitly for the linearely coupled multiplicative noise case,

$$g(p)=\lim_{t\to\infty}\left[\frac{p\omega_0 t+p^2\alpha^2\langle\omega_1^2\rangle\tau_c^2(t/\tau_c-1+e^{-t/\tau_c})}{t}\right] \tag{4.6}$$

$$=p\omega_0+p^2\alpha^2\langle\omega_1^2\rangle\tau_c. \tag{4.7}$$

It follows that

$$g'(0)=\omega_0, \tag{4.8}$$

$$g''(0)=2\alpha^2\langle\omega_1^2\rangle\tau_c. \tag{4.9}$$

As we saw earlier [Eq. (3.31)], for time scales $t\gg\tau_c$, $\alpha^2\langle\omega_1^2\rangle\tau_c$ controls sensitivity to initial conditions and plays therefore, in this respect, the role of the Lyapunov exponent (in the absence of the drift term, $\omega_0=0$). On the other hand, in the framework of the "thermodynamic" formalism $\alpha^2\langle\omega_1^2\rangle\tau_c$ governs the asymptotics of fluctuations whereas the Lyapunov exponent viewed as a long-time average is given by the drift term ω_0 and thus reduces to zero in the case of $\omega_0=0$. There is no contradiction between these results, which merely reflect two different views of the concept of sensitivity to initial conditions. Our view of this concept is more practically oriented and is motivated by the way one monitors deviations between two initially close histories in a physical experiment. The quantity $\langle u_\epsilon(t)\rangle$ that we have introduced for this purpose contains information pertaining to both the mean and the variance of the process and is thus related to $g(p)$ [Eq. (4.1)] rather than to $g'(0)$ or $g''(0)$ [Eqs. (4.4) and (4.5)]. Notice that when the correlation time τ_c gets large, $\alpha^2\langle\omega_1^2\rangle\tau_c$ and thus $g''(0)$ tends to diverge, entailing the divergence of the exponent $g(p)$ for all positive moments [see Eq. (4.7)]. This is the "thermodynamic" signature of the superexponential regime of Eq. (3.32) which, being valid for the transient (though possibly long) period $t\ll\tau_c$, will not show up as such in a theory involving averages over an infinite period of time.

Both Arnold's and our own results show that an exponential sensitivity to intitial conditions (in an averaged sense) can no longer be regarded as the exclusive signature of deterministic chaos. Equally surprising is the fact that certain noise processes (additive noise) give rise to subexponential (power-law) behavior, entailing that in this particular sense the outcome of noise is more predictable than the one of a deterministic chaotic system. Much of the emphasis in the literature placed on the distinction between deterministic motion and noise on this basis therefore appears to lose part of its motivation in the light of our results. In our view the only clear-cut difference that one may still identify pertains to the dimension of the system's invariant set, which in the case of noise increases without bound with increasing embedding dimension.

In future investigations a closer reassessment of the similarities and differences between deterministic and stochastic dynamical systems should be undertaken. It would also be interesting to undertake a study of multiplicative noise in the presence of nonlinear saturation terms enabling the system to reach an invariant probability density for long times.

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